INTUITIONISTIC FUZZY HOPFIELD NEURAL NETWORK AND ITS STABILITY*

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Abstract: Intuitionistic fuzzy sets (IFSs) are generalization of fuzzy sets by adding an additional attribute parameter called non-membership degree. In this paper, a max-min intuitionistic fuzzy Hopfield neural network (IFHNN) is proposed by combining IFSs with Hopfield neural networks. The stability of IFHNN is investigated. It is shown that for any given weight matrix and any given initial intuitionistic fuzzy pattern, the iteration process of IFHNN converges to a limit cycle. Furthermore, under suitable extra conditions, it converges to a stable point within finite iterations. Finally, a kind of Lyapunov stability of the stable points of IFHNN is proved, which means that if the initial state of the network is close enough to a stable point, then the network states will remain in a small neighborhood of the stable point. These stability results indicate the convergence of memory process of IFHNN. A numerical example is also provided to show the effectiveness of the Lyapunov stability of IFHNN

Key words: *Intuitionistic fuzzy sets, intuitionistic fuzzy Hopfield neural network, limit cycle, stable point, Lyapunov stability*

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1. Introduction

In [1–3], Atanassov extends Zadeh's fuzzy sets to intuitionistic fuzzy sets (IFSs) by adding an additional attribute parameter called non-membership degree. IFSs are shown to be superior to fuzzy sets in, for example, semantic expression and inference ability [4]. Various theoretical and applied researches have been performed on IFSs, such as fuzzy topology [5–7], multi-criteria fuzzy decision-making [8–10], clustering [11, 12], medical diagnosis [13, 14] and pattern recognition [15–17].

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Fuzzy neural networks combine fuzzy concepts and fuzzy inference rules with the architecture and learning of neural networks, and have been successfully applied in system identification [18], intelligent control [19–21], pattern classification [22] and expert system [23, 24], etc. Since IFSs have proved to be more powerful to deal with vagueness and uncertainty than fuzzy sets, some researchers have also investigated the combination of IFSs and artificial neural networks [25–29]. In [29], an intuitionistic fuzzy feedforward neural network (IFFFNN) was constructed by combining feedforward neural networks and intuitionistic fuzzy logic, and some operations and two types of transferring functions involved in the working process of IFFFNN were introduced. In this paper, similar to fuzzy Hopfield neural networks [30–33], a max-min intuitionistic fuzzy Hopfield neural network (IFHNN) is proposed by combining IFSs with Hopfield neural networks. The stability of IFHNN is investigated. It is shown that for any given weight matrix and any given initial intuitionistic fuzzy pattern, the iteration process of IFHNN converges to a limit cycle. Furthermore, under suitable extra conditions, it converges to a stable point within finite iterations. Finally, a kind of Lyapunov stability of the stable point of IFHNN is proved, which means that if the initial state of the network is close enough to a stable point, then the network states will remain in a small neighborhood of the stable point. These stability results indicate the convergence of memory process of IFHNN. A numerical example is also provided to show the effectiveness of the Lyapunov stability of IFHNN

The rest of this paper is organized as follows. Some basic concepts of IFSs are collected in Section 2. IFHNN is defined and described in Section 3. A few stability results of IFHNN are given in Section 4. Section 5 presents a numerical example. Some brief conclusions are drawn in Section 6. Finally, proofs of the stability results are provided in an appendix.

2. Preliminaries

Atanassov generalizes Zadeh's fuzzy sets to IFSs:

Definition 1 [1] *Let $\mathbb{X}$ be a given set. An intuitionistic fuzzy set A is an object having the form*

$$A = \{< x, \mu_A(x), \gamma_A(x) > | x \in \mathbb{X}\},$$

where the functions $\mu_A(x) : \mathbb{X} \to [0, 1]$ and $\gamma_A(x) : \mathbb{X} \to [0, 1]$ define the membership degree and the non-membership degree respectively of the element $x \in \mathbb{X}$ to the set A, and for every $x \in \mathbb{X}$, $0 \le \mu_A(x) + \gamma_A(x) \le 1$.

Specifically, when the given set $\mathbb{X}$ is finite, say, $\mathbb{X} = \{x_1, x_2, \ldots, x_m\}$, IFS A can be expressed as a so-called intuitionistic fuzzy vector:

$$A = (< \mu_A(x_1), \gamma_A(x_1) >, < \mu_A(x_2), \gamma_A(x_2) >, \ldots, < \mu_A(x_m), \gamma_A(x_m) >).$$

Definition 2 [2] *Let $A = \{< x, \mu_A(x), \gamma_A(x) > | x \in \mathbb{X}\}$ and $B = \{< x, \mu_B(x), \gamma_B(x) > | x \in \mathbb{X}\}$ be two IFSs. Then, their conjunction, union and complement are defined respectively as*

1) $A \cap B = \{< x, \mu_A(x) \wedge \mu_B(x), \gamma_A(x) \vee \gamma_B(x) > | \forall x \in \mathbb{X}\}$;

2) $A \cup B = \{< x, \mu_A(x) \vee \mu_B(x), \gamma_A(x) \wedge \gamma_B(x) > | \forall x \in \mathbb{X}\}$;
3) $\overline{A} = \{< x, \gamma_A(x), \mu_A(x) > | \forall x \in \mathbb{X}\}$.

Definition 3 [34,35] *Let $\mathbb{X}$ and $\mathbb{Y}$ be two given sets. An intuitionistic fuzzy relation $\mathbf{R}$ from $\mathbb{X}$ to $\mathbb{Y}$ is an IFS of $\mathbb{X}\times\mathbb{Y}$ characterized by the membership function $\mu_{\mathbf{R}}(x,y)$ and the non-membership function $\gamma_{\mathbf{R}}(x,y)$, denoted by*

$$\mathbf{R} = \{< (x,y), \mu_{\mathbf{R}}(x,y), \gamma_{\mathbf{R}}(x,y) > | x \in \mathbb{X},\ y \in \mathbb{Y}\},$$

where the functions $\mu_{\mathbf{R}}(x,y) : \mathbb{X} \times \mathbb{Y} \to [0,1]$ and $\gamma_{\mathbf{R}}(x,y) : \mathbb{X} \times \mathbb{Y} \to [0,1]$ satisfy $0 \le \mu_{\mathbf{R}}(x,y) + \gamma_{\mathbf{R}}(x,y) \le 1$, for every $(x,y) \in \mathbb{X} \times \mathbb{Y}$.

In particular, when the given sets $\mathbb{X}$ and $\mathbb{Y}$ are finite, say, $\mathbb{X} = \{x_1, x_2, \ldots, x_m\}$ and $\mathbb{Y} = \{y_1, y_2, \ldots, y_n\}$, the intuitionistic fuzzy relation $\mathbf{R}$ from $\mathbb{X}$ to $\mathbb{Y}$ can be denoted by an intuitionistic fuzzy matrix $\mathbf{R} = (r_{ij})_{m\times n}$, where $r_{ij} =< \mu_{\mathbf{R}}(x_i, y_j), \gamma_{\mathbf{R}}(x_i, y_j)>$.

Some operations and properties of intuitionistic fuzzy matrixes are defined below (cf. [35, 36]).

Definition 4 *Let $\mathbf{R} = (< \mu_{\mathbf{R}ij}, \gamma_{\mathbf{R}ij} >)_{m\times n}$ and $\mathbf{Q} = (< \mu_{\mathbf{Q}ij}, \gamma_{\mathbf{Q}ij} >)_{m\times n}$ be two intuitionistic fuzzy matrixes. Write $\mathbf{R} \subseteq \mathbf{Q}$, if $\mu_{\mathbf{R}ij} \le \mu_{\mathbf{Q}ij}$ and $\gamma_{\mathbf{R}ij} \ge \gamma_{\mathbf{Q}ij}$, for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.*

Definition 5 *Let $\mathbf{R} = (< \mu_{\mathbf{R}ij}, \gamma_{\mathbf{R}ij} >)_{m\times n}$ and $\mathbf{S} = (< \mu_{\mathbf{S}ij}, \gamma_{\mathbf{S}ij} >)_{n\times l}$ be two intuitionistic fuzzy matrixes. The max-min composite operation "$\circ$" of $\mathbf{R}$ and $\mathbf{S}$ is defined by*

$$\mathbf{R} \circ \mathbf{S} = (< \bigvee_{k=1}^{n} (\mu_{\mathbf{R}ik} \wedge \mu_{\mathbf{S}kj}), \bigwedge_{k=1}^{n} (\gamma_{\mathbf{R}ik} \vee \gamma_{\mathbf{S}kj}) >)_{m\times l}.$$

Property 1 *Let $\mathbf{R}_{m\times n}$, $\mathbf{S}_{n\times l}$ and $\mathbf{T}_{l\times s}$ be intuitionistic fuzzy matrixes. Then the max-min composite operation of intuitionistic fuzzy matrixes satisfies the associative law, i.e., $(\mathbf{R} \circ \mathbf{S}) \circ \mathbf{T} = \mathbf{R} \circ (\mathbf{S} \circ \mathbf{T})$.*

Definition 6 *An intuitionistic fuzzy matrix $\mathbf{R} = (< \mu_{ij}, \gamma_{ij} >)_{n\times n}$ is said to be reflexive, if $\mu_{ii} = 1$ and $\gamma_{ii} = 0$, for $i = 1, 2, \ldots, n$.*

Property 2 *If an intuitionistic fuzzy matrix $\mathbf{R} = (< \mu_{ij}, \gamma_{ij} >)_{n\times n}$ is reflexive, then, there holds $\mathbf{R}^k \subseteq \mathbf{R}^{k+1}$ for $k = 1, 2, \ldots,$ where $\mathbf{R}^{k+1} = \mathbf{R}^k \circ \mathbf{R}$.*

3. Intuitionistic Fuzzy Hopfield Neural Network

Intuitionistic fuzzy Hopfield neural network (IFHNN) is a combination of IFSs and Hopfield neural networks. The basic processing units of IFHNN are intuitionistic fuzzy units, i.e., the input, output and weight signals are all IFSs. In this study, similar to fuzzy Hopfield neural networks [30–33], a max-min IFHNN is constructed. The inner operations involved in the working process of this IFHNN are based on the max-min composite operation mentioned in [29], and the linear transfer function $f(x) = x$ is used as transferring function for the output nodes. The network consists of n processing units which are connected with each other (cf. Fig.1). Both the

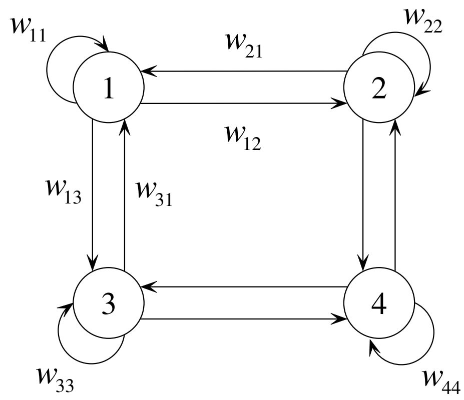

Fig. 1 *Structure of IFHNN for four units.*

input and output signals of the network are intuitionistic fuzzy vectors and the weight matrix is an intuitionistic fuzzy matrix denoted by $\mathbf{W} = (w_{ij})_{n\times n}$, where $w_{ij} =< \mu_{ij}, \gamma_{ij} >$ stands for the weight from the i-th unit to the j-th unit. If the initial state of the network is an intuitionistic fuzzy input pattern

$$X(0) = (< \mu_{X1}(0), \gamma_{X1}(0) >, < \mu_{X2}(0), \gamma_{X2}(0) >, \ldots, < \mu_{Xn}(0), \gamma_{Xn}(0) >),$$

then, the network iteration process is as follows:

$$\begin{aligned} X(t) &= X(t-1) \circ \mathbf{W} \\ &= (< \mu_{X1}(t), \gamma_{X1}(t) >, < \mu_{X2}(t), \gamma_{X2}(t) >, \ldots, < \mu_{Xn}(t), \gamma_{Xn}(t) >), \end{aligned} \quad (1)$$

where

$$\mu_{Xi}(t) = \bigvee_{k=1}^{n} \left(\mu_{Xk}(t-1) \wedge \mu_{ki}\right),$$

$$\gamma_{Xi}(t) = \bigwedge_{k=1}^{n} \left(\gamma_{Xk}(t-1) \vee \gamma_{ki}\right),$$

and $t = 1, 2, \ldots,$ are the discrete time steps. The network will iterate repeatedly according to (1) until a steady state is reached. The final output pattern $X(\infty)$ is taken as an association of the input pattern $X(0)$.

4. Stability Results

Definition 7 [37] *If a sequence of states $\{P_1, P_2, \ldots, P_s\}$ is generated by a feedback network with state transition operator $\mathcal{F}$ such that $\mathcal{F}(P_1) = P_2$, $\mathcal{F}(P_2) = P_3, \ldots, \mathcal{F}(P_k) = P_{k+1}, \ldots, \mathcal{F}(P_s) = P_1$, and there does not exist a subsequence with the same property in this sequence, then this sequence is called a limit cycle and s is called the length of the limit cycle.*

Definition 8 [37] *If there exists a state P of a feedback network such that, for the state transition operator $\mathcal{F}$ of the network, $\mathcal{F}(P) = P$ holds, then P is called a stable point of the network.*

Definition 9 [38–40] *Let $A = (<\mu_{A1}, \gamma_{A1}>, <\mu_{A2}, \gamma_{A2}>, \ldots, <\mu_{An}, \gamma_{An}>)$ and $B = (<\mu_{B1}, \gamma_{B1}>, <\mu_{B2}, \gamma_{B2}>, \ldots, <\mu_{Bn}, \gamma_{Bn}>)$ be two intuitionistic fuzzy patterns. Define the Hamming distance between A and B as*

$$H(A, B) = \frac{1}{2}\sum_{i=1}^{n} \left(|\mu_{Ai} - \mu_{Bi}| + |\gamma_{Ai} - \gamma_{Bi}|\right).$$

Definition 10 [30, 37] *Suppose that the intuitionistic fuzzy pattern P is a stable point of the networks. P is said to be Lyapunov stable, if for any $\varepsilon > 0$, there exists $\delta > 0$, such that for every initial intuitionistic fuzzy pattern X satisfying $H(X, P) < \delta$, $H(X(t), P) < \varepsilon$ holds for $t = 1, 2, \ldots$, where $X(t)$ is the t-th iteration state of the network.*

Now, we are ready to present our main results. Some comments on these theorems can be found in the next section, and the proofs are postponed to the Appendix.

Theorem 1 *For any given intuitionistic fuzzy weight matrix $\mathbf{W}$ and any given initial intuitionistic fuzzy pattern, the iteration process of IFHNN (1) converges to a limit cycle.*

Theorem 2 *Suppose $\mathbf{W}$ is an intuitionistic fuzzy weight matrix of IFHNN with n units. Then, the following statements hold:*

(i) If $\mathbf{W} \subseteq \mathbf{W}^2$, then the iteration process of IFHNN (1) converges to a stable point within finite iterations.

(ii) If $\mathbf{W}$ is reflexive, then the iteration process of IFHNN (1) converges to a stable point within at most $n - 1$ iterations.

Theorem 3 *Suppose intuitionistic fuzzy pattern P is a stable point of IFHNN (1). Then, P is Lyapunov stable.*

5. A Numerical Example

In this section, an illustrative example is given to show the effectiveness of the Lyapunov stability of IFHNN. Suppose the intuitionistic fuzzy weight matrix of IFHNN is

$$\mathbf{W} = \begin{pmatrix} <0.8, 0.1> & <0.4, 0.4> & <0.5, 0.4> & <0.3, 0.5> & <0.1, 0.7> \\ <0.2, 0.5> & <0.9, 0.0> & <0.1, 0.7> & <0.4, 0.4> & <0.7, 0.2> \\ <0.0, 0.6> & <0.0, 0.7> & <0.7, 0.2> & <0.6, 0.3> & <0.4, 0.5> \\ <0.5, 0.4> & <0.4, 0.3> & <0.0, 0.8> & <0.6, 0.2> & <0.2, 0.7> \\ <0.7, 0.2> & <0.6, 0.3> & <0.5, 0.3> & <0.3, 0.5> & <0.8, 0.1> \end{pmatrix}$$

It is easy to verify that $\mathbf{W} \subseteq \mathbf{W}^2$. For the initial intuitionistic fuzzy pattern

$$X(0) = (<0.2, 0.7>, <0.5, 0.2>, <0.3, 0.4>, <0.7, 0.1>, <0.4, 0.5>),$$

the iteration process of IFHNN (1) converges to a stable point

$$X = (< 0.5, 0.2 >, < 0.5, 0.2 >, < 0.5, 0.3 >, < 0.6, 0.2 >, < 0.5, 0.2 >)$$

at the third step.

Next, we consider the Lyapunov stability of the stable point X. For this purpose, we add a random noise in $[-0.001, 0.001]$ to the stable point X and end up with a new initial pattern

$$\begin{aligned}\overline{X}(0) &= (< 0.5009, 0.2007 >, < 0.5001, 0.1995 >, < 0.4993, 0.3006 >, \\ &\quad < 0.5993, 0.1995 >, < 0.4995, 0.2009 >).\end{aligned}$$

Then, the iteration process of IFHNN (1) converges to

$$\begin{aligned}\overline{X} &= (< 0.5009, 0.2000 >, < 0.5001, 0.1995 >, < 0.5000, 0.3000 >, \\ &\quad < 0.5993, 0.2000 >, < 0.5001, 0.2000 >)\end{aligned}$$

at the second step. This shows that when the initial state of the network is close enough to a stable point, the network states remain in a small neighborhood of the stable point.

6. Conclusion

A max-min intuitionistic fuzzy Hopfield neural network (IFHNN) is proposed by combining IFSs with Hopfield neural networks. In addition, the stability of IFHNN is investigated. It is shown that the iteration process of IFHNN always converges to a limit cycle for any given intuitionistic fuzzy weight matrix $\mathbf{W}$ and any given initial intuitionistic fuzzy pattern. In particular, it converges to a stable point within finite iterations if $\mathbf{W} \subseteq \mathbf{W}^2$, and even within $n-1$ iterations if $\mathbf{W}$ is reflexive, where n is the number of the network units. Finally, a kind of Lyapunov stability of the stable point of IFHNN is proved, which means that if the initial state of the network is close enough to a stable point, then the network states will remain in a small neighborhood of the stable point. These stability results indicate the convergence of memory process of IFHNN.

Our work in this paper is preliminary. Investigation on more profound properties and applications of IFHNN might be promising. For instance, in comparison with ordinary Hopfield neural networks, one may consider the following problems: 1) Determine the network weight matrix by using given training patterns. 2) Construct a functional such that the state sequence of the network is a minimization sequence of the functional. 3) Prove more profound convergence theorems of the iteration process. 4) Find practical applications of the network.

Appendix

Proof of Theorem 1: Let $M = \{a_1, a_2, \ldots, a_m\}$ and $Q = \{b_1, b_2, \ldots, b_q\}$ stand for the sets of membership degree and non-membership degree of every element of $\mathbf{W}$, respectively. By the definition of the max-min composite operation "$\circ$", we know that the membership degree and non-membership degree of every element of $\mathbf{W}^k$, for $k = 1, 2, \ldots$, are taken from the set M and Q, respectively. Therefore, there are at most finite different matrixes in the matrix sequence $\{\mathbf{W}^k | k = 1, 2, \ldots, \}$, which means that identical matrixes will appear in the matrix sequence $\{\mathbf{W}^k | k = 1, 2, \ldots, \}$ after finite composite operations. Thus, there exist two positive integers k_0 and k_1, such that $\mathbf{W}^{k_0} = \mathbf{W}^{k_1}$. Assume without loss of generality that $k_0 \leq k_1$. Then, the matrix sequence $\{\mathbf{W}^k | k = 1, 2, \ldots, \}$ converges to the limit cycle $\{\mathbf{W}^{k_0}, \mathbf{W}^{k_0+1}, \ldots, \mathbf{W}^{k_1-1}\}$. Thus, for any initial intuitionistic fuzzy pattern $X(0)$, we have

$$X(k_1) = X(0) \circ \mathbf{W}^{k_1} = X(0) \circ \mathbf{W}^{k_0} = X(k_0),$$

i.e., the iteration process of IFHNN (1) converges to the limit cycle $\{X(k_0), X(k_0 + 1), \ldots, X(k_1 - 1)\}$. Theorem 1 is thus proved. □

Now we are in a position to present two lemmas to be used in our proofs of Theorem 2 and Theorem 3.

Lemma 1 *Given $n \times m$ intuitionistic fuzzy matrixes $\mathbf{A}$, $\mathbf{B}$ and $m \times l$ intuitionistic fuzzy matrixes $\mathbf{C}$, $\mathbf{D}$, if $\mathbf{A} \subseteq \mathbf{B}$ and $\mathbf{C} \subseteq \mathbf{D}$, then $\mathbf{A} \circ \mathbf{C} \subseteq \mathbf{B} \circ \mathbf{D}$.*

Proof. Let $\mathbf{A} = (< \mu_{\mathbf{A}ij}, \gamma_{\mathbf{A}ij} >)$, $\mathbf{B} = (< \mu_{\mathbf{B}ij}, \gamma_{\mathbf{B}ij} >)$, $\mathbf{C} = (< \mu_{\mathbf{C}ij}, \gamma_{\mathbf{C}ij} >)$ and $\mathbf{D} = (< \mu_{\mathbf{D}ij}, \gamma_{\mathbf{D}ij} >)$. It follows from $\mathbf{A} \subseteq \mathbf{B}$, $\mathbf{C} \subseteq \mathbf{D}$ and Definition 4 that $\mu_{\mathbf{A}ij} \leq \mu_{\mathbf{B}ij}$, $\gamma_{\mathbf{A}ij} \geq \gamma_{\mathbf{B}ij}$, $\mu_{\mathbf{C}ij} \leq \mu_{\mathbf{D}ij}$ and $\gamma_{\mathbf{C}ij} \geq \gamma_{\mathbf{D}ij}$ for any i, j. Thus, we have for any i, j

$$\bigvee_{k=1}^{n} (\mu_{\mathbf{A}ik} \wedge \mu_{\mathbf{C}kj}) \leq \bigvee_{k=1}^{n} (\mu_{\mathbf{B}ik} \wedge \mu_{\mathbf{D}kj}) \tag{2}$$

and

$$\bigwedge_{k=1}^{n} (\gamma_{\mathbf{A}ik} \vee \gamma_{\mathbf{C}kj}) \geq \bigwedge_{k=1}^{n} (\gamma_{\mathbf{B}ik} \vee \gamma_{\mathbf{D}kj}). \tag{3}$$

The combination of (2), (3) and Definition 4 leads to $\mathbf{A} \circ \mathbf{C} \subseteq \mathbf{B} \circ \mathbf{D}$. □

Lemma 2 *Assume that $h > 0$, that a_i, $b_i \in [0, 1]$, and that $|a_i - b_i| < h$ for $i = 1, 2, \ldots, n$. Then, the following two inequalities hold:*

$$(a) \left| \bigvee_{i=1}^{n} a_i - \bigvee_{i=1}^{n} b_i \right| < h; \qquad (b) \left| \bigwedge_{i=1}^{n} a_i - \bigwedge_{i=1}^{n} b_i \right| < h.$$

Proof. Inequality (a) has been shown in Lemma 2.2 in [30], and the detail of the proof is omitted.

Next we prove the inequality (b) by induction on n. The inequality (b) is evidently valid for $n = 1$. Let us suppose that (b) is valid for $n = k$, i.e.,

$$\left| \bigwedge_{i=1}^{k} a_i - \bigwedge_{i=1}^{k} b_i \right| = |a - b| < h,$$

where $a = \bigwedge_{i=1}^{k} a_i$ and $b = \bigwedge_{i=1}^{k} b_i$. We proceed to show that (b) is also valid for $n = k + 1$. When $n = k + 1$,

$$\left| \bigwedge_{i=1}^{k+1} a_i - \bigwedge_{i=1}^{k+1} b_i \right| = |a \wedge a_{k+1} - b \wedge b_{k+1}|. \tag{4}$$

We analyze (4) by considering the following four cases.
Case 1 : If $a \geq a_{k+1}$ and $b \geq b_{k+1}$, then

$$|a \wedge a_{k+1} - b \wedge b_{k+1}| = |a_{k+1} - b_{k+1}| < h.$$

Case 2 : If $a \geq a_{k+1}$ and $b < b_{k+1}$, then $|a \wedge a_{k+1} - b \wedge b_{k+1}| = |a_{k+1} - b|$ and $-h < a_{k+1} - b_{k+1} < a_{k+1} - b < a - b < h$. Thus, it is easy to get

$$|a \wedge a_{k+1} - b \wedge b_{k+1}| = |a_{k+1} - b| < h.$$

Case 3 : If $a < a_{k+1}$ and $b < b_{k+1}$, then,

$$|a \wedge a_{k+1} - b \wedge b_{k+1}| = |a - b| < h.$$

Case 4 : If $a < a_{k+1}$ and $b \geq b_{k+1}$, then $|a \wedge a_{k+1} - b \wedge b_{k+1}| = |a - b_{k+1}|$ and $-h < a - b < a - b_{k+1} < a_{k+1} - b_{k+1} < h$. Thus, it is easy to get

$$|a \wedge a_{k+1} - b \wedge b_{k+1}| = |a_{k+1} - b| < h.$$

The above discussions result in, for $n = k + 1$,

$$\left| \bigwedge_{i=1}^{k+1} a_i - \bigwedge_{i=1}^{k+1} b_i \right| = |a \wedge a_{k+1} - b \wedge b_{k+1}| < h.$$

Now we have shown by induction that the inequality (b) always holds. This completes the proof of Lemma 2. $\square$

Proof of Theorem 2: The proof is divided into two parts, dealing with Statements (i) and (ii), respectively.

Proof of Statement (i). Using Lemma 1 and the fact $\mathbf{W} \subseteq \mathbf{W}^2$, we have $\mathbf{W}^k \subseteq \mathbf{W}^{k+1}$ for $k = 1, 2, \ldots$. Therefore, the sequence of membership (resp. non-membership) degree part is monotonically increasing (resp. decreasing) in the intuitionistic fuzzy matrix sequence $\{\mathbf{W}^k | k = 1, 2, \ldots\}$. Notice that there are at most finite different elements in the sequence $\{\mathbf{W}^k | k = 1, 2, \ldots\}$. Hence, there exists a positive integer m such that $\mathbf{W}^m = \mathbf{W}^{m+1}$. For any initial intuitionistic fuzzy pattern $X(0)$, we obtain that

$$X(m) = X(0) \circ \mathbf{W}^m = X(0) \circ \mathbf{W}^{m+1} = X(m + 1).$$

This means that $X(m)$ is a stable point of IFHNN, i.e., the iteration process of IFHNN (1) converges to the stable point $X(m)$ at the m-th iteration. This proves Statement (i).

Proof of Statement (ii). According to the fact that $\mathbf{W}$ is reflexive and noting Property 2, we have for $k = 1, 2, \ldots$,

$$\mathbf{W}^k \subseteq \mathbf{W}^{k+1}. \tag{5}$$

Write $\mathbf{W}^k = (< \mu_{ij}^k, \gamma_{ij}^k >)$. By the definition of the composite operation and the fact that $\mathbf{W}$ is reflexive, it is easy to show that the membership degrees and the non-membership degrees of the diagonal elements of $\mathbf{W}^k$ $(k = 1, 2, \ldots)$ are equal to 1 and 0, respectively, i.e., $\mu_{ii}^k = 1$ and $\gamma_{ii}^k = 0$, for $i = 1, 2, \ldots, n$ and $k = 1, 2, \ldots$.

Next we proceed to discuss the membership and non-membership degrees of the non-diagonal elements of $\mathbf{W}^n$, where n is the number of the network units.

When $i \neq j$,

$$\begin{aligned}\mu_{ij}^n &= \bigvee_{k_1=1}^{n}\left(\mu_{ik_1}\wedge\mu_{k_1j}^{n-1}\right)\\ &= \bigvee_{k_1=1}^{n}\left(\mu_{ik_1}\wedge\left(\bigvee_{k_2=1}^{n}\left(\mu_{k_1k_2}\wedge\mu_{k_2j}^{n-2}\right)\right)\right)\\ &= \bigvee_{k_1=1}^{n}\left(\bigvee_{k_2=1}^{n}\left(\mu_{ik_1}\wedge\mu_{k_1k_2}\wedge\mu_{k_2j}^{n-2}\right)\right).\end{aligned}$$

Then, we can deduce by analogy that

$$\begin{aligned}\mu_{ij}^n &= \bigvee_{1\le k_1,k_2,\dots,k_{n-1}\le n}\left(\mu_{ik_1}\wedge\mu_{k_1k_2}\wedge\cdots\wedge\mu_{k_{n-2}k_{n-1}}\wedge\mu_{k_{n-1}j}\right)\\ &= \max_{1\le k_1,k_2,\dots,k_{n-1}\le n}\min\left(\mu_{ik_1},\mu_{k_1k_2},\cdots,\mu_{k_{n-2}k_{n-1}},\mu_{k_{n-1}j}\right).\end{aligned}$$

Here we have $n+1$ subscripts $i,k_1,k_2,\dots,k_{n-1},j$. Thus, there must exist two subscripts that are equal. We consider the following three cases.
Case 1 : There exists a subscript $k_s \in \{k_1,k_2,\dots,k_{n-1}\}$ such that $k_s = j$. Then,

$$\begin{aligned}\mu_{ij}^n &= \max_{1\le k_1,k_2,\dots,k_{n-1}\le n}\min\left(\mu_{ik_1},\mu_{k_1k_2},\cdots,\mu_{k_{n-2}k_{n-1}},\mu_{k_{n-1}j}\right)\\ &\le \max_{1\le k_1,k_2,\dots,k_{s-1}\le n}\min\left(\mu_{ik_1},\mu_{k_1k_2},\cdots,\mu_{k_{s-1}j}\right)\le\mu_{ij}^{n-1}.\end{aligned}$$

Case 2 : If $i = k_s$, things can be done similarly.
Case 3 : There exist two subscripts $k_s,\ k_r \in \{k_1,k_2,\dots,k_{n-1}\}$, such that $k_s = k_r$ but they are not equal to i or j. Assume without loss of generality that $r > s$. Then,

$$\begin{aligned}\mu_{ij}^n &= \max_{1\le k_1,k_2,\dots,k_{n-1}\le n}\min\left(\mu_{ik_1},\cdots,\mu_{k_{s-1}k_r},\mu_{k_rk_{s+1}},\cdots,\mu_{k_{r-1}k_r},\mu_{k_rk_{r+1}},\cdots,\mu_{k_{n-1}j}\right)\\ &\le \max_{1\le k_1,k_2,\dots,k_{s-1}\le n}\min\left(\mu_{ik_1},\cdots,\mu_{k_{s-1}k_r},\mu_{k_rk_{r+1}},\cdots,\mu_{k_{n-1}j}\right)\le\mu_{ij}^{n-1}.\end{aligned}$$

To sum up, we always have $\mu_{ij}^n \le \mu_{ij}^{n-1}$ for $i,\ j = 1,2,\dots,n$ and $i \neq j$. Analogously, we can prove $\gamma_{ij}^n \ge \gamma_{ij}^{n-1}$ for $i,\ j = 1,2,\dots,n$ and $i \neq j$. Then, we have $\mathbf{W}^n \subseteq \mathbf{W}^{n-1}$. This together with (5) immediately leads to $\mathbf{W}^n = \mathbf{W}^{n-1}$. For any initial intuitionistic fuzzy pattern $X(0)$ we obtain that

$$X(n-1) = X(0)\circ\mathbf{W}^{n-1} = X(0)\circ\mathbf{W}^n = X(n).$$

This means that $X(n-1)$ is a stable point of IFHNN, i.e., the iteration process of IFHNN (1) converges to a stable point within at most $(n-1)$ iterations. Now Statement (ii) is proved. And this completes the proof of Theorem 2.

Proof of Theorem 3: Let $P = (<\mu_{P1}, \gamma_{P1}>, <\mu_{P2}, \gamma_{P2}>, \dots, <\mu_{Pn}, \gamma_{Pn}>)$ be a stable point of IFHNN, the network weight matrix be $\mathbf{W}$, and the t-th iteration state of the network be

$$X(t) = (<\mu_{X1}(t), \gamma_{X1}(t)>, <\mu_{X2}(t), \gamma_{X2}(t)>, \dots, <\mu_{Xn}(t), \gamma_{Xn}(t)>),$$

where $t = 0, 1, 2\dots$ and $X(0)$ is the initial intuitionistic fuzzy pattern. Write $\mathbf{W}^t = (<\mu_{ij}^t, \gamma_{ij}^t>)_{n\times n}$. Then, $X(t) = X(0) \circ \mathbf{W}^t$, i.e., for $j = 1, 2, \dots, n$,

$$\mu_{Xj}(t) = \bigvee_{i=1}^{n} \left(\mu_{Xi}(0) \wedge \mu_{ij}^t\right),\ \gamma_{Xj}(t) = \bigwedge_{i=1}^{n} \left(\gamma_{Xi}(0) \vee \gamma_{ij}^t\right).$$

Noticing that P is a stable point of IFHNN, we have $P = P \circ \mathbf{W}^t$, i.e., for $j = 1, 2, \dots, n$,

$$\mu_{Pj} = \bigvee_{i=1}^{n} \left(\mu_{Pi} \wedge \mu_{ij}^t\right),\ \gamma_{Pj} = \bigwedge_{i=1}^{n} \left(\gamma_{Pi} \vee \gamma_{ij}^t\right).$$

For given $\varepsilon > 0$, we choose $\delta = \varepsilon/2n$. For any initial intuitionistic fuzzy pattern $X(0)$ satisfying $H(X(0), P) < \delta$, we have for $i = 1, 2, \dots, n$,

$$|\mu_{Xi}(0) - \mu_{Pi}| < \varepsilon/n,\ |\gamma_{Xi}(0) - \gamma_{Pi}| < \varepsilon/n.$$

Then, for $i = 1, 2, \dots, n$,

$$\begin{aligned} & |\mu_{Xi}(0) \wedge \mu_{ij}^t - \mu_{Pi} \wedge \mu_{ij}^t| \\ = & \left|\frac{\mu_{Xi}(0) + \mu_{ij}^t - |\mu_{Xi}(0) - \mu_{ij}^t|}{2} - \frac{\mu_{Pi} + \mu_{ij}^t - |\mu_{Pi} - \mu_{ij}^t|}{2}\right| \\ = & \frac{\left|\mu_{Xi}(0) - \mu_{Pi} + |\mu_{Pi} - \mu_{ij}^t| - |\mu_{Xi}(0) - \mu_{ij}^t|\right|}{2} \\ \leq & |\mu_{Xi}(0) - \mu_{Pi}| < \varepsilon/n. \end{aligned} \tag{6}$$

Analogously, it is easy to obtain that, for $i = 1, 2, \dots, n$,

$$|\gamma_{Xi}(0) \vee \gamma_{ij}^t - \gamma_{Pi} \vee \gamma_{ij}^t| \leq |\gamma_{Xi}(0) - \gamma_{Pi}| < \varepsilon/n. \tag{7}$$

According to (6), (7) and Lemma 2, we have for $j = 1, 2, \dots, n$,

$$|\mu_{Xj}(t) - \mu_{Pj}| = \left|\bigvee_{i=1}^{n} \left(\mu_{Xi}(0) \wedge \mu_{ij}^t\right) - \bigvee_{i=1}^{n} \left(\mu_{Pi} \wedge \mu_{ij}^t\right)\right| < \varepsilon/n,$$

$$|\gamma_{Xj}(t) - \gamma_{Pj}| = \left|\bigwedge_{i=1}^{n} \left(\gamma_{Pi} \vee \gamma_{ij}^t\right) - \bigwedge_{i=1}^{n} \left(\gamma_{Pi} \vee \gamma_{ij}^t\right)\right| < \varepsilon/n.$$

Hence,

$$H(X(t), P) = \frac{1}{2}\sum_{j=1}^{n} \left[|\mu_{Xj}(t) - \mu_{Pj}| + |\gamma_{Xj}(t) - \gamma_{Pj}|\right] < \varepsilon.$$

This completes the proof of Theorem 3.

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