EXPONENTIAL STABILITY OF PERIODIC SOLUTIONS FOR INERTIAL COHEN-GROSSBERG-TYPE NEURAL NETWORKS

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Abstract: In this paper, the exponential stability of periodic solutions for inertial Cohen-Grossberg-type neural networks are investigated. First, by properly chosen variable substitution the system is transformed to first order differential equation. Second, some sufficient conditions which can ensure the existence and exponential stability of periodic solutions for the system are obtained by using constructing suitable Lyapunov function and differential mean value theorem, applying the analysis method and inequality technique. Finally, two examples are given to illustrate the effectiveness of the results.

Key words: *Inertial Cohen-Grossberg-type neural networks, Lyapunov function, inequality technique, periodic solutions, exponential stability*

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1. Introduction

In recent decades, much attention has been devoted to the studies of artificial neural networks partially due to the fact that neural networks can be applied to signal processing, image processing, pattern recognition, control and optimization problems. The Cohen-Grossberg neural network [3], proposed in 1983, is focal research subject. There are many interesting phenomena in the dynamical behaviors of Cohen-Grossberg neural network. In the past years, the stability and periodic solutions problem for a class of Cohen-Grossberg neural networks

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -\alpha_i(x_i(t)) \left(h_i(x_i(t)) - \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) - \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_{ij})) + I_i(t) \right), \tag{1}$$

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has received much research attention, and many good results related to this problem have been reported, see [2, 3, 14, 15, 17].

On the other hand, some authors studied neural networks added the inertia and obtained some results. For example, Li et al. [9] added the inertia to a delay differential equation which can be described by

$$\ddot{x} = a\dot{x} - bx + cf(x - hx(t - \tau)).$$

and obtained obvious chaotic behavior. Liu et al. [12, 13] found chaotic behavior of the inertial two-neuron system with time through numerical simulation, and gave that the system will lose its stability when the time delay is increased and will rise a quasi-periodic motion and chaos under the interaction of the periodic excitation. Wheeler and Schieve [16] added the inertia to a continuous-time, Hopfield effective-neuron system which is shown to exhibit chaos. They explained the chaos is confirmed by Lyapunov exponents, power spectra, and phase space plots, this system is described by

$$\begin{aligned} \ddot{x_1} &= -a_{11}\dot{x_1} - a_{12}x_1 + a_{13}\tanh(x_1) + a_{14}\tanh(x_2), \\ \ddot{x_2} &= -b_{11}\dot{x_2} - b_{12}x_2 + b_{13}\tanh(x_1) + b_{14}\tanh(x_2). \end{aligned}$$

Babcocka et al. [1] studied the electronic neural networks with added inertia and found when the neuron couplings are of an inertial nature, the dynamics can be complex, in contrast to the simpler behavior displayed when they are the standard resistor-capacitor variety. For various values of the neuron gain and the quality factor of the couplings they found ringing about the stationary points, instability and spontaneous oscillation, intertwined basins of attraction, and chaotic response to a harmonic drive. Juhong and Jing [5] considered an inertial four-neuron delayed bidirectional associative memory model. Weak resonant double Hopf bifurcations are completely analyzed in the parameter space of the coupling weight and the coupling delay by the perturbation-incremental scheme. In [4], author investigated a kinematical description of traveling waves in the oscillations in the networks is extended of the networks with inertia. When the inertia is below a critical value and the state of each neuron is over-damped, properties of the networks are qualitatively the same as those without inertia. The duration of the transient oscillations increases with inertia, and the increasing rate of the logarithm of the duration becomes more than double. When the inertia exceeds a critical value and the state of each neuron becomes under-damped, properties of the networks qualitatively change. The periodic solution is stabilized through the pitchfork bifurcation as inertia increases. More bifurcations occur so that various periodic solutions are generated, and the stability of the periodic solutions changes alternately. Further, stable oscillations generated with inertia are observed in an experiment on an analog circuit. From the above, the inertia can be considered a useful tool which is added to help in the generation of chaos in neural systems.

Others, Liu et al. [10, 11] investigated the Hopf bifurcation and dynamics of an inertial two-neuron system or a single inertial neuron mode. Zhao et. al. [18] investigated the stability and the bifurcation of a class of inertial neural networks. The authors Ke and Miao [6, 7, 8] investigated stability of equilibrium point and periodic solutions in inertial BAM neural networks with time delays and unbounded delays, and the stability of inertial Cohen-Grossberg-type neural networks with time delays, respectively.

In this paper, we consider the following inertial Cohen-Grossberg-type neural networks with time delays

$$\begin{aligned}\frac{\mathrm{d}^2x_i(t)}{\mathrm{d}t^2} = &- \beta_i\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} - \alpha_i(x_i(t))(h_i(x_i(t)) - \sum_{j=1}^{n} a_{ij}f_j(x_j(t)) \\ &- \sum_{j=1}^{n} b_{ij}f_j(x_j(t-\tau_{ij})) + I_i(t)),\end{aligned} \tag{2}$$

for $i = 1, 2, \dots, n$, where the second derivative are called an inertial term of system (2), $\beta_i > 0$ are constants, $x_i(t)$ denotes the states variable of the i-th neuron at the time t, $\alpha_i(\cdot)$ denotes an amplification function; $h_i(\cdot)$ is the behaved function, a_{ij} and b_{ij} are connection weights of the neural networks; f_j denotes the activation function of j-th neuron at the time t; τ_{ij} is time delay of j-th neuron at the time t and satisfies $0 \le \tau_{ij} \le \tau$; $I_i(t)$ denotes the external inputs on the i-th neuron at the time t.

The initial values of system (2) are

$$x_i(s) = \varphi_i(s), \qquad \frac{\mathrm{d}x_i(s)}{\mathrm{d}s} = \psi_i(s), \qquad -\tau \le s \le 0, \tag{3}$$

where $\varphi_i(s)$, $\psi_i(s)$ are bounded and continuous functions.

From the viewpoints of mathematics and physics, the system (2) is a class of nonlinear second-order dynamical system where $\alpha_i > 0$ is a damping coefficient, then the system (1) can be considered as a model overdamped (i.e. the damp tend to infinite). But in some practical problems, we need to consider the existence and stability of the system when it has damping (or low damping). For example, pendulum equation with dissipation term

$$\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = -\alpha\frac{\mathrm{d}x(t)}{\mathrm{d}t} - \beta x - \gamma\sin(t),$$

and forced Duffing equation

$$\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = -\alpha\frac{\mathrm{d}x(t)}{\mathrm{d}t} - x(\beta x + \gamma x^2) + \delta\cos(vt),$$

which have applied background.

This paper is organized as follows. Some preliminaries are given in Section 2. The sufficient conditions are derived to ensure the existence and exponential stability of periodic solutions for inertial Cohen-Grossberg-type neural networks in Section 3. Two illustrative examples are given to show the effectiveness of the proposed theory in Section 4.

2. Preliminaries

Throughout this paper, we make the following assumptions

H1 For each $i = 1, 2, \dots, n$, functions $\alpha_i(x)$ are differentiable and satisfy $|\alpha_i'(x)| \le \bar{A}_i$, and $0 < \underline{\alpha}_i \le \alpha_i(x) \le \bar{\alpha}_i$ for all $x \in \Re$.

H2 For each $i = 1, 2, \dots, n$, functions $h_i(x)$ are differentiable and satisfy $0 < \underline{h}_i \le h_i'(x) \le \bar{h}_i$, for all $x \in \Re$.

H3 For each $j = 1, 2, \dots, n$, the activation functions f_j satisfy Lipschitz condition, and there exist constant $l_j > 0, \bar{f}_j > 0$, such that

$$|f_j(v_1) - f_j(v_2))| \le l_j|v_1 - v_2|, \quad |f_j(x)| \le \bar{f}_j,$$

for $v_1, v_2, x \in \Re$.

H4 For each $i = 1, 2, \dots, n$, $I_i(t)$ are continuously periodic functions defined on $t \in [0, \infty)$ with common period $\omega > 0$, and satisfy $0 < \underline{I}_i \le I_i(t) \le \bar{I}_i$.

H5 Let $g_i(x) = \alpha_i(x)h_i(x)$, for each $i = 1, 2, \dots, n$, there exist constant $T_i > 0$ and $K_i > 0$, such that

$$0 < T_i \le g_i'(x) \le K_i, \quad x \in \Re.$$

Introducing variable transformation

$$y_i(t) = \frac{\mathrm{d}x_i(t)}{\mathrm{d}t} + x_i(t), \quad i = 1, 2, \dots, n,$$

then (2) and (3) can be rewritten as

$$\begin{cases} \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -x_i(t) + y_i(t), \\ \dfrac{\mathrm{d}y_i(t)}{\mathrm{d}t} = -\,(1-\beta_i)x_i(t) - (\beta_i - 1)y_i(t) \\ \qquad - \alpha_i(x_i(t)) \left(h_i(x_i(t)) - \displaystyle\sum_{j=1}^{n} a_{ij} f_j(x_j(t)) - \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_{ij})) + I_i(t) \right) \end{cases} \tag{4}$$

and

$$\begin{cases} x_i(s) = \varphi_i(s), \\ \frac{\mathrm{d}x_i(s)}{\mathrm{d}s} = \psi_i(s), \qquad -\tau \le s \le 0. \\ y_i(s) = \varphi_i(s) + \psi_i(s), \end{cases} \tag{5}$$

Definition 1. *Let* $x^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t))^T$ *be an* ω*- periodic solution of system* (2) *with initial value*

$$x_i^*(s) = \varphi_i^*(s), \quad \frac{dx_i(s)}{ds} = \psi_i^*(s), \quad -\tau \le s \le 0.$$

If there exist constants $\delta > 0$ and $M > 0$, such that for every solution $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ of system (2) *with any initial value*

$$x_i(s) = \varphi_i(s), \quad \frac{dx_i(s)}{ds} = \psi_i(s), \quad -\tau \le s \le 0,$$

$$\sum_{i=1}^{n} |x_i(t) - x_i^*(t)|^2 \le Me^{-\delta t}\|\varphi - \varphi^*\|^2, \quad t > 0,$$

then $x^*(t)$ is said to be globally exponentially stable, where

$$\|\varphi - \varphi^*\|^2 = \sup_{-\tau\le t\le 0} \sum_{i=1}^{n} |\varphi_i(t) - \varphi_i^*(t)|^2.$$

3. Main results

In this section, we will derive some sufficient conditions which can ensure the existence and exponential stability of periodic solutions for the system (2).

Theorem 1. *For system* (2)*, under the hypotheses H1-H4, then $x_i(t)$, $\frac{dx_i(t)}{dt}$ are bounded, $i = 1, 2, \ldots, n$, $t \ge 0$.*

Proof. It follows from (2) that

$$\begin{aligned}
\frac{\mathrm{d}^2 x_i(t)}{\mathrm{d}t^2} &= -\beta_i \frac{\mathrm{d}x_i(t)}{\mathrm{d}t} - sgn(x_i(t))\alpha_i(x_i(t))\cdot \qquad (6)\\
&\quad \cdot \left[h_i(x_i(t)) - \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) - \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_{ij})) + I_i(t)\right]\\
&= -\beta_i \frac{\mathrm{d}x_i(t)}{\mathrm{d}t} - sgn(x_i(t))\alpha_i(x_i(t))\\
&\quad \cdot \left[h_i(x_i(t)) - h_i(0) + h_i(0) - \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) - \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_{ij})) + I_i(t)\right]\\
&\le -\beta_i \frac{\mathrm{d}x_i(t)}{\mathrm{d}t} - \underline{\alpha}_i \underline{h}_i |x_i(t)| + \bar{\alpha}_i \left(|h_i(0)| + \sum_{j=1}^{n} |a_{ij}|\bar{f}_j + \sum_{j=1}^{n} |b_{ij}|\bar{f}_j + \bar{I}_i\right).
\end{aligned}$$

From (6), we can obtain

$$|x_i(t)| \le C_1 \mathrm{e}^{\lambda_1 t} + C_2 e^{\lambda_2 t} + \frac{\bar{\alpha}_i}{\underline{\alpha}_i \underline{h}_i}\left[|h_i(0)| + \sum_{j=1}^{n} |a_{ij}|\bar{f}_j + \sum_{j=1}^{n} |b_{ij}|\bar{f}_j + \bar{I}_i\right], \qquad (7)$$

where $\lambda_{1,2} = \frac{-\beta_i \pm \sqrt{\beta_i^2 - 4\underline{\alpha}_i \underline{h}_i}}{2}$, C_1, C_2 are any real constants.

Since $\beta_i > 0$, we have $Re(\lambda_1) < 0$, $Re(\lambda_2) < 0$, formula (7) shows that all solutions $x_i(t)$ to (2) are bounded for $i = 1, 2, \ldots, n, t \geq 0$.

On the other hand, from (2) we also can obtain

$$\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = e^{-\beta_i t}\frac{\mathrm{d}x_i(0)}{\mathrm{d}t} - e^{-\beta_i t}\int_0^t e^{\beta_i s}\alpha_i(x_i(s))[h_i(x_i(s)) - \sum_{j=1}^n a_{ij}f_j(x_j(s))$$

$$-\sum_{j=1}^n b_{ij}f_j(x_j(s-\tau_{ij})) + I_i(s)]ds, \quad i = 1, 2, \ldots, n. \tag{8}$$

From the above we can see, $x_i(t)$ are bounded, we may assume that $|x_i(t)| \leq R_i$, $R_i > 0$ are constants, $i = 1, 2, \ldots, n$. From (8), we have

$$|\frac{\mathrm{d}x_i(t)}{\mathrm{d}t}| \leq |\psi_i(0)| + \bar{\alpha}_i\left[\bar{h}_i R_i + |h_i(0)| + \sum_{j=1}^n |a_{ij}|\bar{f}_j + \sum_{j=1}^n |b_{ij}|\bar{f}_j + \bar{I}_i\right]. \tag{9}$$

Formula (9) shows that all solutions $\frac{\mathrm{d}x_i(t)}{\mathrm{d}t}$ are bounded for $i = 1, 2, \ldots, n, t \geq 0$. □

Theorem 2. *Under the hypotheses H1-H5, if* $\beta_i - K_i > 0$ *and*

$$-2 - T_i + \beta_i + \bar{A}_i\bar{I}_i + \bar{A}_i\sum_{j=1}^n \bar{f}_j\left[|a_{ij}| + |b_{ij}|\right] + \sum_{j=1}^n \bar{\alpha}_j[|a_{ji}| + |b_{ji}|]l_i < 0,$$

$$2 - \beta_i - T_i + \bar{A}_i\bar{I}_i + \bar{A}_i\sum_{j=1}^n \bar{f}_j\left[|a_{ij}| + |b_{ij}|\right] + \bar{\alpha}_i\sum_{j=1}^n[|a_{ij}| + |b_{ij}|]l_j < 0,$$

for $i = 1, 2, \ldots, n$, *then system* (2) *has one* ω*-periodic solution, which is globally exponentially stable.*

Proof. Let $\bar{x}(t) = (\bar{x}_1(t), \bar{x}_2(t), \ldots, \bar{x}_n(t))^T$ be an solution of system (2) with initial value

$$\bar{x}_i(s) = \bar{\varphi}_i(s), \quad \frac{\mathrm{d}x_i(s)}{\mathrm{d}s} = \bar{\psi}_i(s), \quad -\tau \leq s \leq 0,$$

$x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ be an solution of system (2) with initial value

$$x_i(s) = \varphi_i(s), \quad \frac{\mathrm{d}x_i(s)}{\mathrm{d}s} = \psi_i(s), \quad -\tau \leq s \leq 0.$$

Let

$$\varphi_i^*(s) = \varphi_i(s) + \psi_i(s), \quad \bar{\varphi}_i^*(s) = \bar{\varphi}_i(s) + \bar{\psi}_i(s), \quad -\tau \leq s \leq 0,$$
$$\bar{y}_i(t) = \bar{x}_i(t) + \bar{x}_i'(t), \quad z_i(t) = x_i(t) - \bar{x}_i(t), \quad v_i(t) = y_i(t) - \bar{y}_i(t).$$

From (4), we can obtain

$$\begin{cases} \dfrac{\mathrm{d}z_i(t)}{\mathrm{d}t} = -z_i(t) + v_i(t), \\ \dfrac{\mathrm{d}v_i(t)}{\mathrm{d}t} = -(1-\beta_i)z_i(t) - (\beta_i - 1)v_i(t) + \alpha_i(x_i(t)) \left[\sum\limits_{j=1}^{n} a_{ij}(f_j(x_j(t)) - f_j(\bar{x}_j(t))) \right. \\ \quad \left. + \sum\limits_{j=1}^{n} b_{ij}(f_j(x_j(t-\tau_{ij})) - f_j(\bar{x}_j(t-\tau_{ij})))\right] \\ \quad +(\alpha_i(x_i(t)) - \alpha_i(\bar{x}_i(t))) \left[\sum\limits_{j=1}^{n} a_{ij} f_j(\bar{x}_j(t)) + \sum\limits_{j=1}^{n} b_{ij} f_j(\bar{x}_j(t-\tau_{ij})) - I_i(t)\right] \\ \quad -[\alpha_i(x_i(t))h_i(x_i(t)) - \alpha_i(\bar{x}_i(t))h_i(\bar{x}_i(t))]. \end{cases} \tag{10}$$

Since functions $\alpha_i(x)$ and $h_i(x)$ are differentiable, using differential mean value theorem, we have

$$\begin{aligned} \alpha_i(x_i(t)) - \alpha_i(\bar{x}_i(t)) &= \alpha_i'(\xi_i)z_i(t), \\ \alpha_i(x_i(t))h_i(x_i(t)) - \alpha_i(\bar{x}_i(t))h_i(\bar{x}_i(t)) = g_i(x_i) - g_i(\bar{x}_i) &= g_i'(\bar{\xi}_i)z_i(t), \end{aligned}$$

where ξ_i and $\bar{\xi}_i$ lie between x_i and $\bar{x}_i$.

Since $0 < T_i \le g_i'(x) \le K_i$, if $\beta_i - K_i > 0$, then $\beta_i - g_i'(\xi_i) \ge \beta_i - K_i > 0$, and $0 < \beta_i - g_i'(\xi_i) \le \beta_i - T_i$.

From (10) we get

$$\frac{\mathrm{d}z_i^2(t)}{\mathrm{d}t} = -z_i^2(t) + z_i(t)v_i(t). \tag{11}$$

$$\begin{aligned} \frac{\mathrm{d}v_i^2(t)}{\mathrm{d}t} &= -(1-\beta_i)z_i(t)v_i(t) - (\beta_i - 1)v_i^2(t) \\ &\quad +v_i(t)\alpha_i(x_i(t))[\sum_{j=1}^{n} a_{ij}(f_j(x_j(t)) - f_j(\bar{x}_j(t))) \\ &\quad + \sum_{j=1}^{n} b_{ij}(f_j(x_j(t-\tau_{ij})) - f_j(\bar{x}_j(t-\tau_{ij})))] \\ &\quad +v_i(t)(\alpha_i(x_i(t)) - \alpha_i(\bar{x}_i(t)))[\sum_{j=1}^{n} a_{ij} f_j(\bar{x}_j(t)) \\ &\quad + \sum_{j=1}^{n} b_{ij} f_j(\bar{x}_j(t-\tau_{ij})) - I_i(t)] - v_i(t)[\alpha_i(x_i(t))h_i(x_i(t)) - \alpha_i(\bar{x}_i(t))h_i(\bar{x}_i(t))] \\ &= -(1-\beta_i)z_i(t)v_i(t) - (\beta_i - 1)v_i^2(t) + v_i(t)\alpha_i(x_i(t))[\sum_{j=1}^{n} a_{ij}(f_j(x_j(t)) - f_j(\bar{x}_j(t))) \\ &\quad + \sum_{j=1}^{n} b_{ij}(f_j(x_j(t-\tau_{ij})) - f_j(\bar{x}_j(t-\tau_{ij})))] + \alpha_i'(\xi_i)z_i(t)v_i(t)[\sum_{j=1}^{n} a_{ij} f_j(\bar{x}_j(t)) \\ &\quad + \sum_{j=1}^{n} b_{ij} f_j(\bar{x}_j(t-\tau_{ij})) - I_i(t)] - g_i'(\bar{\xi}_i)z_i(t)v_i(t). \end{aligned} \tag{12}$$

From (11) and (12), we can obtain

$$\frac{1}{2}\frac{\mathrm{d}(z_i^2(t)+v_i^2(t)))}{\mathrm{d}t} \leq -z_i^2(t) + [\beta_i - g_i'(\bar{\xi}_i)]z_i(t)v_i(t) - (\beta_i - 1)v_i^2(t)$$
$$+\bar{\alpha}_i[\sum_{j=1}^n |a_{ij}|l_j|z_j(t)| + \sum_{j=1}^n |b_{ij}|l_j|z_j(t-\tau_{ij})|]|v_i(t)|$$
$$+\bar{A}_i[\sum_{j=1}^n |a_{ij}|\bar{f}_j + \sum_{j=1}^n |b_{ij}|\bar{f}_j + \bar{I}_i]|z_i(t)||v_i(t)|$$
$$\leq -z_i^2(t) + [\beta_i - T_i]\frac{(z_i^2(t)+v_i^2(t))}{2} - (\beta_i - 1)v_i^2(t)$$
$$+\bar{\alpha}_i[\sum_{j=1}^n |a_{ij}|l_j\frac{(z_j^2(t)+v_i^2(t))}{2} + \sum_{j=1}^n |b_{ij}|l_j\frac{(z_j^2(t-\tau_{ij})+v_i^2(t))}{2}]$$
$$+\bar{A}_i[\sum_{j=1}^n |a_{ij}|\bar{f}_j + \sum_{j=1}^n |b_{ij}|\bar{f}_j + \bar{I}_i]\frac{(z_i^2(t)+v_i^2(t))}{2}$$
$$\leq \tfrac{1}{2}[-2 - T_i + \beta_i + \bar{A}_i\sum_{j=1}^n \bar{f}_j(|a_{ij}|+|b_{ij}|) + \bar{A}_i\bar{I}_i]z_i^2(t) + \tfrac{1}{2}[2-\beta_i - T_i$$
$$+\bar{\alpha}_i\sum_{j=1}^n l_j(|a_{ij}|+|b_{ij}|) + \bar{A}_i\sum_{j=1}^n \bar{f}_j(|a_{ij}|+|b_{ij}|)$$
$$+\bar{A}_i\bar{I}_i]v_i^2(t) + \bar{\alpha}_i\sum_{j=1}^n \tfrac{|a_{ij}|}{2}l_jz_j^2(t) + \bar{\alpha}_i\sum_{j=1}^n \tfrac{|b_{ij}|}{2}l_jz_j^2(t-\tau_{ij}). \tag{13}$$

We consider the Lyapunov function:

$$V(t) = \sum_{i=1}^n \left[\frac{z_i^2(t)+v_i^2(t)}{2}\,\mathrm{e}^{\varepsilon t} + \bar{\alpha}_i\sum_{j=1}^n \frac{|b_{ij}|}{2}l_j\int_{t-\tau_{ji}}^t \mathrm{e}^{\varepsilon(s+\tau_{ji})}z_j^2(s)\,\mathrm{d}s\right] \tag{14}$$

$\varepsilon > 0$ is a small number.

Calculating the upper right Dini-derivative $D^+V(t)$ of $V(t)$ along the solution of (10), using (13) we have

$$D^+V(t) = \sum_{i=1}^n\{\varepsilon\frac{z_i^2(t)+v_i^2(t)}{2}\,\mathrm{e}^{\varepsilon t} + \frac{1}{2}\frac{\mathrm{d}z_i^2(t)}{\mathrm{d}t} + v_i^2(t))\,\mathrm{e}^{\varepsilon t}$$
$$+\bar{\alpha}_i\sum_{j=1}^n \frac{|b_{ij}|}{2}l_j[z_j^2(t)e^{\varepsilon(t+\tau_{ji})} - z_j^2(t-\tau_{ji})e^{\varepsilon t}]\}$$
$$\leq \mathrm{e}^{\varepsilon t}\sum_{i=1}^n\{\varepsilon\frac{z_i^2(t)+v_i^2(t)}{2} + \frac{1}{2}[-2-T_i+\beta_i+\bar{A}_i\sum_{j=1}^n \bar{f}_j(|a_{ij}|+|b_{ij}|)+\bar{A}_i\bar{I}_i]z_i^2(t)$$
$$+\frac{1}{2}[2-\beta_i-T_i+\bar{\alpha}_i\sum_{j=1}^n l_j(|a_{ij}|+|b_{ij}|) + \bar{A}_i\sum_{j=1}^n \bar{f}_j(|a_{ij}|+|b_{ij}|) + \bar{A}_i\bar{I}_i]v_i^2(t)$$
$$+\bar{\alpha}_i\sum_{j=1}^n \frac{|a_{ij}|}{2}l_jz_j^2(t) + \bar{\alpha}_i\sum_{j=1}^n \frac{|b_{ij}|}{2}l_jz_j^2(t-\tau_{ij})$$
$$+\bar{\alpha}_i\sum_{j=1}^n \frac{|b_{ij}|}{2}l_j[z_j^2(t)e^{\varepsilon\tau} - z_j^2(t-\tau_{ji})]\}$$
$$\leq \tfrac{\mathrm{e}^{\varepsilon t}}{2}\sum_{i=1}^n\{[\varepsilon - 2 - T_i + \beta_i + \bar{A}_i\bar{I}_i + \bar{A}_i\sum_{j=1}^n \bar{f}_j(|a_{ij}|+|b_{ij}|)$$
$$+\sum_{j=1}^n \bar{\alpha}_j(|a_{ji}| + \mathrm{e}^{\varepsilon\tau}|b_{ji}|)l_i]z_i^2(t) + [\varepsilon + 2 - \beta_i - T_i + \bar{A}_i\bar{I}_i)$$
$$+\bar{A}_i\sum_{j=1}^n \bar{f}_j(|a_{ij}|+|b_{ij}|) + \bar{\alpha}_i\sum_{j=1}^n(|a_{ij}|+|b_{ij}|)l_j]v_i^2(t)\}. \tag{15}$$

From condition of Theorem 2, we can choose a small $\varepsilon > 0$ such that

$$\varepsilon - 2 - T_i + \beta_i + \bar{A}_i\bar{I}_i + \bar{A}_i\sum_{j=1}^{n}\bar{f}_j(|a_{ij}| + |b_{ij}|) + \sum_{j=1}^{n}\bar{\alpha}_j(|a_{ji}| + e^{\varepsilon\tau}|b_{ji}|)l_i \le 0,$$

$$\varepsilon + 2 - \beta_i - T_i + \bar{A}_i\bar{I}_i + \bar{A}_i\sum_{j=1}^{n}\bar{f}_j(|a_{ij}| + |b_{ij}|) + \bar{\alpha}_i\sum_{j=1}^{n}(|a_{ij}| + |b_{ij}|)l_j \le 0,$$

for $i = 1, 2 \ldots, n$.
From (15), we get $D^+V(t) \le 0$, and so $V(t) \le V(0)$, for all $t \ge 0$.
From (14), we have

$$V(t) \ge \sum_{i=1}^{n}\frac{z_i^2(t) + v_i^2(t)}{2}\,e^{\varepsilon t} = \sum_{i=1}^{n}\frac{e^{\varepsilon t}}{2}[(x_i(t) - \bar{x}_i(t))^2 + (y_i(t) - \bar{y}_i(t))^2]. \quad (16)$$

$$\begin{aligned}
V(0) &= \sum_{i=1}^{n}\{\frac{z_i^2(0) + v_i^2(0)}{2} + \bar{\alpha}_i\sum_{j=1}^{n}\frac{|b_{ij}|}{2}l_j\int_{-\tau_{ij}}^{0} z_j^2(s)\,e^{\varepsilon(s+\tau_{ij})}ds\}\\
&= \sum_{i=1}^{n}\{\frac{(\varphi_i(0) - \bar{\varphi}_i(0))^2}{2} + \frac{(\varphi_i^*(0) - \bar{\varphi}_i^*(0))^2}{2}\\
&\quad + \bar{\alpha}_i\sum_{j=1}^{n}\frac{|b_{ij}|}{2}l_j\int_{-\tau_{ij}}^{0}(\varphi_j(s) - \bar{\varphi}_j(s))^2\,e^{\varepsilon(s+\tau_{ij})}ds\}\\
&\le \frac{\|\varphi - \bar{\varphi}\|^2}{2} + \frac{\|\varphi^* - \bar{\varphi}^*\|^2}{2} + \tau\sum_{j=1}^{n}\max_{1\le i\le n}\{\bar{\alpha}_i\frac{|b_{ij}}{2}|l_i\}e^{\varepsilon\tau}\|\varphi - \bar{\varphi}\|^2\\
&\le \frac{1}{2}[1 + \tau\sum_{j=1}^{n}\max_{1\le i\le n}\{\bar{\alpha}_i|b_{ij}|l_j\}\,e^{\varepsilon\tau}]\|\varphi - \bar{\varphi}\|^2 + \frac{1}{2}\|\varphi^* - \bar{\varphi}^*\|^2. \quad (17)
\end{aligned}$$

Since $V(0) \ge V(t)$, from (16) and (17), we obtain

$$\begin{aligned}
&\sum_{i=1}^{n}\frac{e^{\varepsilon t}}{2}[(x_i(t) - \bar{x}_i(t))^2 + (y_i(t) - \bar{y}_i(t))^2]\\
&\qquad \le \frac{1}{2}[1 + \tau\sum_{j=1}^{n}\max_{1\le i\le n}\{|b_{ij}|l_j\}\,e^{\varepsilon\tau}]\|\varphi - \bar{\varphi}\|^2 + \tfrac{1}{2}\|\varphi^* - \bar{\varphi}^*\|^2. \quad (18)
\end{aligned}$$

By multiplying both sides of (18) with $2e^{-\varepsilon t}$, we get

$$\sum_{i=1}^{n}[(x_i(t) - \bar{x}_i(t))^2 + (y_i(t) - \bar{y}_i(t))^2] \le M\,e^{-\varepsilon t}\|\varphi - \bar{\varphi}\|^2, \quad t > 0. \quad (19)$$

for all $t \ge 0$, where $M = \left\{1 + \tau\sum_{j=1}^{n}\max_{1\le i\le n}\{|b_{ij}|l_j\}\,e^{\varepsilon\tau} + \frac{\|\varphi^* - \bar{\varphi}^*\|^2}{\|\varphi - \bar{\varphi}\|^2}\right\}$.

For $i = 1, 2, \ldots, n$, when $I_i(t)$ are continuously periodic functions defined on $t \in [0, \infty)$ with common period $\omega > 0$, if $x_i(t)$ are the solutions of (2), then for any natural number k, $x_i(t + k\omega)$ are the solutions of (2). Thus, from (19) there exist constants $N > 0$ and $\delta > 0$, such that

$$|x_i(t + (k+1)\omega) - x_i(t + k\omega)| \le N\,e^{-\delta(t+k\omega)}\|\varphi - \bar{\varphi}\|^2. \quad (20)$$

It is noted that for any natural number p

$$x_i(t+(p+1)\omega) = x_i(t) + \sum_{k=0}^{p}(x_i(t+(k+1)\omega) - x_i(t+k\omega)).$$

Thus

$$|x_i(t+(p+1)\omega)| \le |x_i(t)| + \sum_{k=0}^{p}|x_i(t+(k+1)\omega) - x_i(t+k\omega)|. \qquad (21)$$

Since $x_i(t)$ are bounded, it follows (20) and (21) that $\{x(t+p\omega)\}$ uniformly converges to a continuous function $x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))$ on any compact set of $\Re$. When $x_i(t)$ and $\frac{\mathrm{d}x_i(t)}{\mathrm{d}t}$ are bounded, the same can be proved that $\{y(t+p\omega)\}$ uniformly converges to a continuous function $y^*(t) = (y_1^*(t), y_2^*(t), \ldots, y_n^*(t))$ on any compact set of $\Re$.

Now we will show that $x^*(t)$ is the ω- *periodic* solution of system (2).

First, $x^*(t)$ is ω- *periodic* function, since

$$x^*(t+\omega) = \lim_{p\to\infty} x(t+(p+1)\omega) = x^*(t).$$

Second, we prove that $x^*(t)$ is a solution of system (2). In fact, since

$$\begin{cases} \dfrac{\mathrm{d}x_i(t+p\omega)}{\mathrm{d}t} = -x_i(t+p\omega) + y_i(t+p\omega), \\ \dfrac{\mathrm{d}y_i(t+p\omega)}{\mathrm{d}t} = -(1-\beta_i)x_i(t+p\omega) - (\beta_i - 1)y_i(t+p\omega) \\ \qquad -\alpha_i(x_i(t+p\omega))[h_i(x_i(t+p\omega)) - \sum\limits_{j=1}^{n} a_{ij} f_j(x_j(t+p\omega)) \\ \qquad - \sum\limits_{j=1}^{n} b_{ij} f_j(x_j(t+p\omega - \tau_{ij})) + I_i(t+p\omega)]. \end{cases} \qquad (22)$$

Since $\{x(t+p\omega)\}$ uniformly converges to a continuous function $x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))$, and that $\{y(t+p\omega)\}$ uniformly converges to a continuous function $y^*(t) = (y_1^*(t), y_2^*(t), \ldots, y_n^*(t))$, under the hypotheses $(H1) - (H5)$, (22) implies that

$$\left\{\frac{\mathrm{d}x_i(t+p\omega)}{\mathrm{d}t}\right\}, \quad \left\{\frac{\mathrm{d}y_i(t+p\omega)}{\mathrm{d}t}\right\},$$

uniformly converges to a continuous function on any compact set of $\Re$, respectively. Thus, letting $p \to \infty$, we obtain

$$\begin{cases} \dfrac{\mathrm{d}x_i^*(t)}{\mathrm{d}t} = -x_i^*(t) + y_i^*(t), \\ \dfrac{\mathrm{d}y_i^*(t)}{\mathrm{d}t} = -(1-\beta_i)x_i^*(t) - (\beta_i - 1)y_i^*(t) - \alpha_i(x_i^*(t))[h_i(x_i^*(t)) \\ \qquad - \sum\limits_{j=1}^{n} a_{ij} f_j(x_j^*(t)) - \sum\limits_{j=1}^{n} b_{ij} f_j(x_j^*(t-\tau_{ij})) + I_i(t)]. \end{cases} \qquad (23)$$

It means that is $x^*(t)$ is a periodic solution of system (2). From (19), we have

$$\sum_{i=1}^{n}(x_i(t) - x_i^*(t))^2 \le M\mathrm{e}^{-\varepsilon t}\|\varphi - \varphi^*\|^2, \quad t > 0,$$

then it is globally exponentially stable. □

Theorem 3. *Under the hypotheses H1-H5, there is one ω-periodic solution of system* (2)*, which is globally exponentially stable, if following conditions hold*

$$\beta_i - 1 - k_i > 0, \quad -T_i + \bar{A}_i \sum_{j=1}^{n} \bar{f}_j(|a_{ij}| + |b_{ij}|) + \bar{A}_i \bar{I}_i + \bar{\alpha}_i \sum_{j=1}^{n} l_j(|a_{ij}| + |b_{ij}|) < 0,$$

for $i = 1, 2 \ldots, n$.

Proof. Let $\bar{x}(t) = (\bar{x}_1(t), \bar{x}_2(t), \ldots, \bar{x}_n(t))^T$ be an solution of system (2) with initial value

$$\bar{x}_i(s) = \bar{\varphi}_i(s), \quad \frac{\mathrm{d}\bar{x}_i(s)}{\mathrm{d}s} = \bar{\psi}_i(s), \quad -\tau \le s \le 0,$$

$x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ be an solution of system (2) with initial value

$$x_i(s) = \varphi_i(s), \quad \frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = \psi_i(s), \quad -\tau \le s \le 0.$$

From (10), we can obtain

$$\frac{\mathrm{d}|z_i(t)|}{\mathrm{d}t} = \mathrm{sgn}(z_i(t))(-z_i(t) + v_i(t)) \le -|z_i(t)| + |v_i(t)|. \tag{24}$$

$$\begin{aligned}
\frac{\mathrm{d}|v_i(t)|}{\mathrm{d}t} &= \mathrm{sgn}(v_i(t))\{-(1-\beta_i)z_i(t) - (\beta_i - 1)v_i(t) \\
&+\alpha_i(x_i(t))[\sum_{j=1}^{n} a_{ij}(f_j(x_j(t)) - f_j(\bar{x}_j(t))) \\
&+\sum_{j=1}^{n} b_{ij}(f_j(x_j(t-\tau_{ij})) - f_j(\bar{x}_j(t-\tau_{ij})))] + (\alpha_i(x_i(t)) - \alpha_i(\bar{x}_i(t)))[\sum_{j=1}^{n} a_{ij} f_j(\bar{x}_j(t)) \\
&+\sum_{j=1}^{n} b_{ij} f_j(\bar{x}_j(t-\tau_{ij})) - I_i(t)] - [\alpha_i(x_i(t))h_i(x_i(t)) - \alpha_i(\bar{x}_i(t))h_i(\bar{x}_i(t))]\} \\
&= \mathrm{sgn}(v_i(t))\{-(1-\beta_i)z_i(t) - (\beta_i - 1)v_i(t) \\
&+\alpha_i(x_i(t))[\sum_{j=1}^{n} a_{ij}(f_j(x_j(t)) - f_j(\bar{x}_j(t))) \\
&+\sum_{j=1}^{n} b_{ij}(f_j(x_j(t-\tau_{ij})) - f_j(\bar{x}_j(t-\tau_{ij})))] + \alpha_i'(\xi_i)z_i(t)[\sum_{j=1}^{n} a_{ij} f_j(\bar{x}_j(t)) \\
&+\sum_{j=1}^{n} b_{ij} f_j(\bar{x}_j(t-\tau_{ij})) - I_i(t)] - g_i'(\bar{\xi}_i)z_i(t)\} \\
&\le (\beta_i - 1 - g_i'(\bar{\xi}_i))|z_i(t)| - (\beta_i - 1)|v_i(t)| + \bar{\alpha}_i \sum_{j=1}^{n} l_j[|a_{ij}||z_j(t)| + |b_{ij}||z_j(t-\tau_{ij})|] \\
&+\bar{A}_i[\sum_{j=1}^{n} \bar{f}_j(|a_{ij}| + |b_{ij}|) + \bar{I}_i]|z_i(t)| \\
&\le [\beta_i - 1 - T_i + \bar{A}_i \sum_{j=1}^{n} \bar{f}_j(|a_{ij}| + |b_{ij}|) + \bar{A}_i\bar{I}_i]|z_i(t)| - (\beta_i - 1)|v_i(t)| \\
&+\bar{\alpha}_i \sum_{j=1}^{n} l_j[|a_{ij}||z_j(t)| + |b_{ij}||z_j(t-\tau_{ij})|].
\end{aligned} \tag{25}$$

From (24) and (25), we can obtain

$$|z_i(t)| \le e^{-t}|z_i(0)| + \int_0^t e^{s-t}|v_i(s)|ds. \tag{26}$$

$$|v_i(t)| \leq e^{(1-\beta_i)t}|v_i(0)|$$
$$+ [\beta_i - 1 - T_i + \bar{A}_i \sum_{j=1}^{n} \bar{f}_j(|a_{ij}| + |b_{ij}|) + \bar{A}_i \bar{I}_i] \int_0^t e^{(\beta_i-1)(s-t)}|z_i(s)|ds$$
$$+ \bar{\alpha}_i \sum_{j=1}^{n} l_j[|a_{ij}| \int_0^t e^{(\beta_i-1)(s-t)}|z_j(s)|ds$$
$$+|b_{ij}| \int_0^t e^{(\beta_i-1)(s-t)}|z_j(s-\tau_{ij})|ds]. \quad (27)$$

We consider the functions $g_i(\xi)$ given by

$$g_i(\xi) = \xi - T_i + \bar{A}_i \sum_{j=1}^{n} \bar{f}_j(|a_{ij}|+|b_{ij}|) + \bar{A}_i \bar{I}_i + \bar{\alpha}_i \sum_{j=1}^{n} l_j(|a_{ij}|+|b_{ij}|e^{\xi\tau}), i = 1, 2, \ldots, n.$$

Obviously

$$\frac{\mathrm{d}g_i(\xi)}{\mathrm{d}\xi} > 0, \quad \lim_{\xi \to +\infty} g_i(\xi) = +\infty, \quad g_i(0) < 0,$$

for $i = 1, 2, \ldots, n$.

Therefore, there exist constants $\xi_i \in (0, +\infty)$, such that

$$g_i(\xi_i) = 0, \quad i = 1, 2, \ldots, n.$$

We choose $\bar{\xi} = min\{\xi_1, \xi_2, \ldots, \xi_n\}$, then $\bar{\xi} > 0$, when $0 < \sigma < \bar{\xi}$, we have

$$\sigma - T_i + \bar{A}_i \sum_{j=1}^{n} \bar{f}_j(|a_{ij}| + |b_{ij}|) + \bar{A}_i \bar{I}_i + \bar{\alpha}_i \sum_{j=1}^{n} l_j(|a_{ij}| + |b_{ij}|e^{\sigma\tau}) < 0.$$

Since the initial values $\varphi_i(s), \psi_i(s)$ are bounded and continuous functions, then exist $N_1, N_2 > 0$, such that $|\varphi_i(t)| \leq N1, |\psi_i(t)| \leq N_2$, $1, 2, \ldots, n, t \in [-\tau, 0]$. Let $L = N_1 + N_2$, we will show that for any sufficiently small constant $\varepsilon > 0$,

$$|z_i(t)| < (L+\varepsilon)\mathrm{e}^{-\sigma t}, \ |v_i(t)| < (L+\varepsilon)e^{-\sigma t}, t \geq 0, i = 1, 2, \ldots, n. \quad (28)$$

If (28) does not hold, there exists some $k \in \{1, 2, \ldots, n\}$ and $t_1 \geq 0$, such that

$$\begin{cases} |z_k(t_1)| = (L+\varepsilon)\mathrm{e}^{-\sigma t_1}, \ \frac{\mathrm{d}^+|z_k(t_1)|}{\mathrm{d}t} \geq 0, \\ |z_i(t)| < (L+\varepsilon)\mathrm{e}^{-\sigma t}, \ t \in [0, t_1), \\ |v_i(t)| < (L+\varepsilon)\mathrm{e}^{-\sigma t}, \ t \in [0, t_1]. \end{cases} \quad (29)$$

or

$$\begin{cases} |z_k(t_1)| = |v_k(t_1)| = (L+\varepsilon)\mathrm{e}^{-\sigma t_1}, \\ |z_i(t)| < (L+\varepsilon)\mathrm{e}^{-\sigma t}, \ t \in [0, t_1), \\ |v_i(t)| < (L+\varepsilon)\mathrm{e}^{-\sigma t}, \ t \in [0, t_1). \end{cases} \quad (30)$$

or

$$\begin{cases} |v_k(t_1)| = (L+\varepsilon)\mathrm{e}^{-\sigma t_1}, \\ |v_i(t)| < (L+\varepsilon)\mathrm{e}^{-\sigma t}, \ t \in [0, t_1), \\ |z_i(t)| < (L+\varepsilon)\mathrm{e}^{-\sigma t}, \ t \in [0, t_1]. \end{cases} \quad (31)$$

Therefore, by (24) and (29), we have

$$\frac{\mathrm{d}^+|z_k(t_1)|}{\mathrm{d}t} \leq -|z_k(t_1)| + |v_k(t_1)| < -(L+\varepsilon)\mathrm{e}^{-\sigma t_1} + (L+\varepsilon)e^{-\sigma t_1} = 0,$$

which is a contradiction. By (27) and (30)(or (31)), we obtain

$$\begin{aligned}
|v_k(t_1)| = (L+\varepsilon)\mathrm{e}^{-\sigma t_1} \le e^{(1-\beta_k)t_1}|v_k(0)| + [\beta_k - 1 - T_k \\
+\bar{A}_k \sum_{j=1}^{n} \bar{f}_j(|a_{kj}| + |b_{kj}|) + \bar{A}_k\bar{I}_k] \int_0^{t_1} \mathrm{e}^{(\beta_k-1)(s-t_1)}|z_k(s)|ds \\
+\bar{\alpha}_i \sum_{j=1}^{n} l_j[|a_{kj}| \int_0^{t_1} \mathrm{e}^{(\beta_k-1)(s-t_1)}|z_j(s)|ds + |b_{kj}| \int_0^{t_1} \mathrm{e}^{(\beta_k-1)(s-t_1)}|z_j(s-\tau_{kj})|ds] \\
\le (L+\varepsilon)\{\mathrm{e}^{(1-\beta_k)t_1} + [\beta_k - 1 - T_k + \bar{A}_k \sum_{j=1}^{n} \bar{f}_j(|a_{kj}| + |b_{kj}|) \\
+\bar{A}_k\bar{I}_k] \int_0^{t_1} \mathrm{e}^{(\beta_k-1)(s-t_1)-\sigma s}ds + \bar{\alpha}_i \sum_{j=1}^{n} l_j[|a_{kj}| \int_0^{t_1} \mathrm{e}^{(\beta_k-1)(s-t_1)-\sigma s}ds \\
+|b_{kj}|e^{\tau\sigma} \int_0^{t_1} \mathrm{e}^{(\beta_k-1)(s-t_1)-\sigma s}ds]\} \\
\le (L+\varepsilon)\{\mathrm{e}^{(1-\beta_k)t_1} + [\beta_k - 1 - T_k + \bar{A}_k \sum_{j=1}^{n} \bar{f}_j(|a_{kj}| + |b_{kj}|) \\
+\bar{A}_k\bar{I}_k + \bar{\alpha}_i \sum_{j=1}^{n} l_j(|a_{kj}| + |b_{kj}|\mathrm{e}^{\tau\sigma})] \frac{\mathrm{e}^{-\sigma t_1} - e^{(1-\beta_k)t_1}}{\beta_k - 1 - \sigma}\} \\
\le (L+\varepsilon)\mathrm{e}^{-\sigma t_1}\{\mathrm{e}^{(1-\beta_k+\sigma)t_1} + [\beta_k - 1 - T_k + \bar{A}_k \sum_{j=1}^{n} \bar{f}_j(|a_{kj}| + |b_{kj}|) \\
+\bar{A}_k\bar{I}_k + \bar{\alpha}_i \sum_{j=1}^{n} l_j(|a_{kj}| + |b_{kj}|\mathrm{e}^{\tau\sigma})] \frac{1 - \mathrm{e}^{(1-\beta_k+\sigma)t_1}}{\beta_k - 1 - \sigma}\}. \qquad (32)
\end{aligned}$$

Since

$$\beta_k - 1 - \sigma > \beta_k - 1 - T_k + \bar{A}_k \sum_{j=1}^{n} \bar{f}_j(|a_{kj}| + |b_{kj}|) + \bar{A}_k\bar{I}_k + \bar{\alpha}_i \sum_{j=1}^{n} l_j(|a_{kj}| + |b_{kj}|\mathrm{e}^{\tau\sigma}),$$

we have

$$\begin{aligned}
[\beta_k - 1 - T_k + \bar{A}_k \sum_{j=1}^{n} \bar{f}_j(|a_{kj}| + |b_{kj}|) + \bar{A}_k\bar{I}_k \\
+\bar{\alpha}_i \sum_{j=1}^{n} l_j(|a_{kj}| + |b_{kj}|\mathrm{e}^{\tau\sigma})]/(\beta_k - 1 - \sigma) < 1.
\end{aligned}$$

From (32), we have

$$L + \varepsilon < L + \varepsilon,$$

which is a contradiction. Thus (28) holds, let $\varepsilon \to 0$, we have

$$|z_i(t)| \le L\mathrm{e}^{-\sigma t}, \ \ |v_i(t)| \le L\mathrm{e}^{-\sigma t}, \ \ 1, 2, \ldots, n, \ \ t > 0. \qquad (33)$$

From (33), there exist constants $\sigma > 0$ and $M > 0$ such that

$$\sum_{i=1}^{n} |x_i(t) - \bar{x}_i(t)|^2 \le M\mathrm{e}^{-\sigma t}\|\varphi - \bar{\varphi}\|^2, \ \ 1, 2, \ldots, n, \ \ t > 0. \qquad (34)$$

Next, similar to the methods of the proof of Theorem 2, we can proof that system (2) has one ω-*periodic* solution which is globally exponentially stable. □

4. Numerical examples

In this Section, we give two examples to show showing our results.

Example 1. *We consider the following inertial Cohen-Grossberg-type neural networks*

$$\begin{aligned}\frac{d^2x_i(t)}{dt^2} =& -\beta_i\frac{dx_i(t)}{dt} - \alpha_i(x_i(t))(h_i(x_i(t)) - \sum_{j=1}^{2} a_{ij}f_j(x_j(t)) \\ & - \sum_{j=1}^{2} b_{ij}f_j(x_j(t-\tau_{ij})) + I_i(t)),\end{aligned} \tag{35}$$

for $i = 1, 2,$ *where*

$$\beta_1 = 2.2,\ \ \beta_2 = 2.1,\ a_{11} = 0.2,\ a_{12} = 0.3,\ a_{21} = -0.2,\ a_{22} = 0.1,$$

$$b_{11} = 0.1, b_{12} = 0.2,\ b_{21} = -0.1, b_{22} = -0.2,\ \alpha_i(x) = 1 + \frac{1}{1+x^2},$$

$$h_i(x) = x,\quad f_i(x) = \frac{1}{16}\sin(x),\ I_i(t) = \frac{1}{6}(2+\sin(t)).$$

Obviously,

$$1 \le \alpha_i(x) \le 2,\ \ |\alpha_i'(x)| = |\frac{-2x}{(1+x^2)^2}| \le 1,\ \ \frac{1}{6} < I_i(t) < \frac{1}{2},\ \ h_i'(x) = 1,$$

$$g_i(x) = \alpha_i(x)h_i(x) = x + \frac{x}{1+x^2},\ \ \frac{7}{8} \le g_i'(x) = 1 + \frac{1-x^2}{(1+x^2)^2} \le 2.$$

$$|f_i(x) - f_i(y)| \le |x-y|/16,\ \ i = 1, 2.$$

We select

$$\omega = 2\pi,\ \bar{f}_i = \frac{1}{16},\ \underline{\alpha}_i = 1,\ \bar{\alpha}_i = 2,\ \bar{A}_i = 1,\ \underline{h}_i = 1,\ \bar{h}_i = 1,\ \bar{I}_i = \frac{1}{2},$$

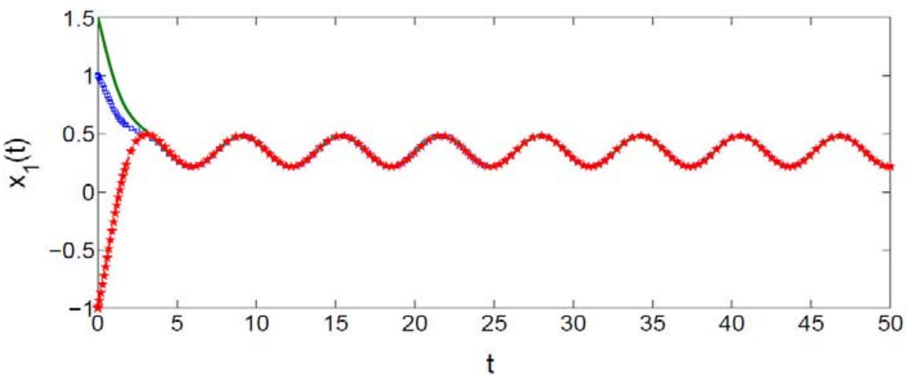

Fig. 1 *Transient response of state variables $x_1(t)$ of Example 1.*

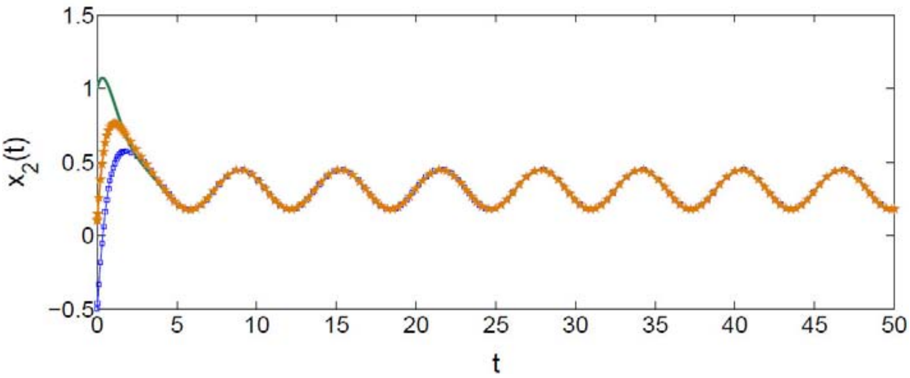

Fig. 2 *Transient response of state variables $x_2(t)$ of Example 1.*

$$T_i = \frac{7}{8},\ K_i = 2,\ l_i = 1/16,\ i = 1, 2.$$

Thus, hypotheses $(H1) - (H5)$ are hold.

For numerical simulation, let $\tau_{11} = 0.1$, $\tau_{12} = 0.2$, $\tau_{21} = 0.2$, $\tau_{22} = 0.1$, the following any three cases are given:

$$[\varphi_1(0), \varphi_2(0), \psi_1(0), \psi_2(0)] = [1; -0.1; 0.8; 1.3]; \quad [1.5; 0.8; 1.2; 1.5]; \quad [0.1; 1.2; 1.4; 1.8].$$

Figs. 1 and Figs. 2 depict the time responses of state variables of $x_1(t)$ and $x_2(t)$ of system in Example 1, respectively.

On the other hand, we have the following results by simple calculation

$$-2 - T_1 + \beta_1 + \bar{A}_1 \bar{I}_1 + \bar{A}_1 \sum_{j=1}^{2} \bar{f}_j [|a_{1j}| + |b_{1j}|] + \sum_{j=1}^{2} \bar{\alpha}_j [|a_{j1}| + |b_{j1}|] l_1 = -\frac{1}{20} < 0,$$

$$-2 - T_2 + \beta_2 + \bar{A}_2 \bar{I}_2 + \bar{A}_2 \sum_{j=1}^{2} \bar{f}_j [|a_{2j}| + |b_{2j}|] + \sum_{j=1}^{2} \bar{\alpha}_j [|a_{j2}| + |b_{j2}|] l_2 = -\frac{11}{80} < 0,$$

$$2 - \beta_1 - T_1 + \bar{A}_1 \bar{I}_1 + \bar{A}_1 \sum_{j=1}^{2} \bar{f}_j [|a_{1j}| + |b_{1j}|] + \bar{\alpha}_1 \sum_{j=1}^{2} [|a_{1j}| + |b_{1j}|] l_j = -\frac{9}{20} < 0,$$

$$2 - \beta_2 - T_2 + \bar{A}_2 \bar{I}_2 + \bar{A}_2 \sum_{j=1}^{2} \bar{f}_j [|a_{2j}| + |b_{2j}|] + \bar{\alpha}_2 \sum_{j=1}^{2} [|a_{2j}| + |b_{2j}|] l_j = -\frac{27}{80} < 0,$$

and $\beta_1 - K_1 = 0.2 > 0, \beta_2 - K_2 = 0.1 > 0$.

Then, the conditions of Theorem 2 hold. From Theorem 2 that this system (35) has one 2π- periodic solution, and all other solutions of system (35) exponentially converge to it as $t \to +\infty$. Evidently, this consequence is coincident with the results of numerical simulation.

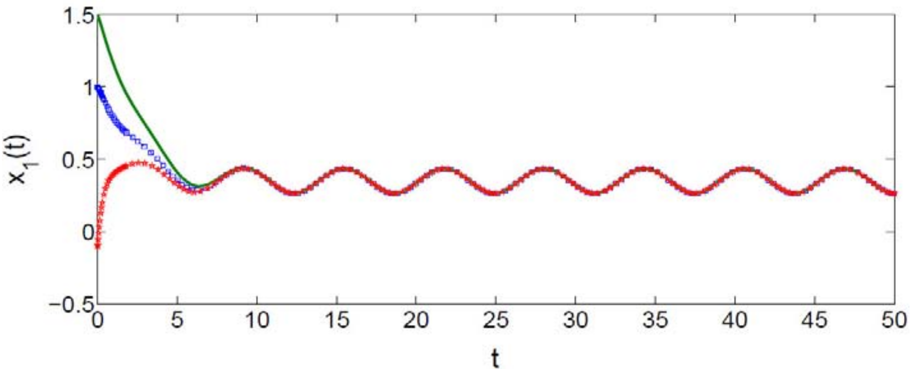

Fig. 3 *Transient response of state variables $x_1(t)$ of Example 2.*

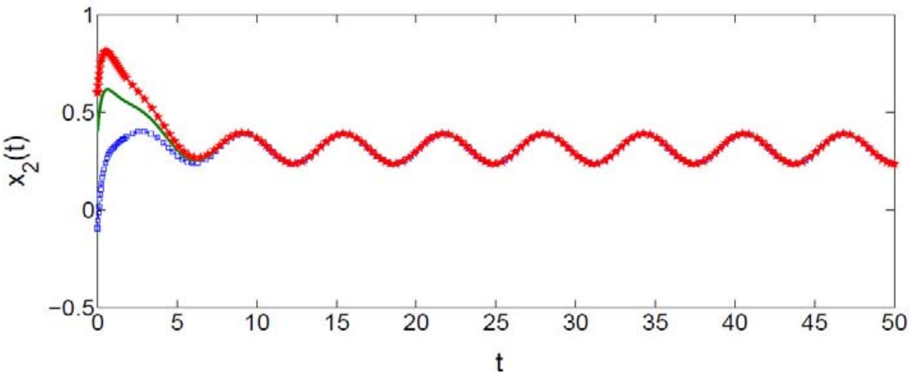

Fig. 4 *Transient response of state variables $x_2(t)$ of Example 2.*

Example 2. *For system* (35)*, we let* $\beta_1 = 3.5$*,* $\beta_2 = 4$*, the other parameters are the same as that in Example 1.*

For numerical simulation, the following any three cases are given:

$[\varphi_1(0), \varphi_2(0), \psi_1(0), \psi_2(0)] = [1; -0.5; 0.8; 1.3]; \quad [1.5; 1; 1; 1.5]; \quad [-1; 0.1; -0.4; 1.8].$

Figs. 3 and Figs. 4 depict the time responses of state variables of $x_1(t)$ and $x_2(t)$ of system in Example 2, respectively.

On the other hand, we have the following results by simple calculation

$$\beta_1 - 1 - K_1 = 0.5 > 0, \ \ \beta_2 - 1 - K_2 = 1 > 0,$$

$$-T_1 + \bar{A}_1 \bar{I}_1 + \bar{A}_1 \sum_{j=1}^{2} \bar{f}_j [|a_{1j}| + |b_{1j}|] + \bar{\alpha}_1 \sum_{j=1}^{2} l_j [|a_{1j}| + |b_{1j}|] = -0.225 < 0,$$

$$-T_2 + \bar{A}_2\bar{I}_2 + \bar{A}_2\sum_{j=1}^{2}\bar{f}_j[|a_{2j}| + |b_{2j}|] + \bar{\alpha}_2\sum_{j=1}^{2}l_j[|a_{2j}| + |b_{2j}|] = -0.3 < 0.$$

Then, the conditions of Theorem 3 hold. From Theorem 3 that this system (35) *has one* 2π*- periodic solution, and all other solutions of system* (35) *exponentially converge to it as* $t \to +\infty$*. Evidently, this consequence is coincident with the results of numerical simulation.*

Remark 1. *Example 1 and Example 2 showed system* (35) *has one* ω*-periodic solution, which is globally exponentially stable. In Example 1, there is*

$$\beta_1 - 1 - k_1 = -0.8 < 0.$$

But this condition isn't satisfied Theorem 3. While in Example 2, there is

$$-2 - T_1 + \beta_1 + \bar{A}_1\bar{I}_1 + \bar{A}_1\sum_{j=1}^{2}\bar{f}_j[|a_{1j}| + |b_{1j}|] + \sum_{j=1}^{2}\bar{\alpha}_j[|a_{j1}| + |b_{j1}|]l_1 = 0.25 > 0.$$

This condition isn't satisfied Theorem 2. It showed that Theorem 2 and Theorem 3 have different applications.

In fact, the parameter β_i *in Theorem 2 must be satisfy* $\beta_i > K_i,\ \ 2 - T_i < \beta_i < 2 + T_i$*. For Theorem 3 it is only required to satisfy* $\beta_i > 1 + K_i$*.*
Therefore, Theorems 2 and Theorem 3 can solve different problems.

5. Conclusion

In this paper, we give theorems to ensure the existence and the exponential stability of the periodic solution for inertial Cohen-Grossberg-type neural networks. Novel existence and stability conditions are stated with simple algebraic forms and their verification and applications are straightforward and convenient. Especially, we give different conditions in Theorems 2 and Theorems 3 to ensure the exponential stability of the periodic solution, which have different advantages in different problems and applications. Finally two examples illustrate the effectiveness in different conditions.

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